Cost Minimization of a Competitive Firm

Pahlaj Moolio*, and Jamal Nazrul Islam**

ABSTRACT

One of the economists' missions is to predict the behavioral responses of consumers or firms on the assumption that optimizing continues. Once this capability is developed, economists try to manage "today" to optimize future economic return of the inputs. Techniques to predict future performance vary from an educated guess based on an appropriate analogy to very complex analytical and numerical calculations and approximations. However, what they all have in common is that they analyze performance in past to say something to obtain constrained optimal output in future. Considering Lagrange multiplier technique applied to a firm's cost minimization problem subject to production function as an output constraint, an attempt has been made in this paper to apply necessary and sufficient conditions for optimal values. We gave interpretation of Lagrange multiplier and showed that its value is positive. Examining the behavior of the firm; that is, if the cost of a particular input increases, the firm needs to consider decreasing level of that particular input; at the same time, there is no effect on the level of other inputs; also that when the demand of product increases, the firm should consider increasing its level of inputs: capital, labour and other inputs, have been derived.

JEL. Classification: L11; L23; L25; C51; C61; C67

Key words: Lagrange Multiplier, Optimization, Cost Minimization, Cobb-Douglas Production Function.

1. INTRODUCTION

---

* Ph.D. Fellow, Research Centre for Mathematical and Physical Sciences, University of Chittagong, Chittagong, Bangladesh; and Professor and Academic Advisor, Pannasastra University of Cambodia, Phnom Penh, Cambodia. Corresponding Author: Pahlaj Moolio. E-mail: pahlajmoolio@gmail.com

** [1]Emeritus Professor, Research Centre for Mathematical and Physical Sciences, University of Chittagong, Chittagong, Bangladesh.

Acknowledgements: Author would like to thank the editor and anonymous referees for their comments and insight in improving the draft copy of this article. Author furthur would like to declare that this manuscript is Original and has not previously been published, and that it is not currently on offer to another publisher; and transfer copy rights to the publisher of this journal.

Date of Recieving: 15-05-2008; **Accepted:** 20-07-2008; **Published**: 01-09-2008

The method of Lagrange multipliers has been used to facilitate the determination of necessary conditions; normally, this method was considered as a device for transforming a constrained problem to a higher dimensional unconstrained problem (Islam 1997). Using Lagrange multiplier method, Baxley and Moorhouse (1984) analyzed an example of utility maximization, and provided a formulation for nontrivial constrained optimization problem with special reference to application in economics. They considered implicit functions with assumed characteristic qualitative features and provided illustration of an example, generating meaningful economic behavior. This approach and formulation may enable one to view optimization problems in economics from a somewhat wider perspective. In the end of their paper, they suggested several other types of problems from economics. Taking into account Cobb-Douglas production function in two variables (factors: capital and labour), Pahlaj (2002) studied the behavior of the firm, by considering one of those suggested problems.

Developing mathematical model in section 2 and considering an explicit form of Cobb-Douglas production function in three variables (factors: capital, labour, and other inputs) as an output constraint, we apply necessary conditions section 3, and find stationary point and optimal value of the cost function. Applying sufficient conditions to cost minimization problem in section 5, we extend the work of Pahlaj (2002). In section 4, we give reasonable interpretation of the Lagrange multiplier in the context of this particular problem, besides using it as a device for transforming a constrained problem into a higher dimensional unconstrained problem. In section 6, analyzing comparative static results (Chiang 1984) with the application of Implicit Function Theorem, we examined the behavior of the firm, that is, how a change in the input costs will affect the situation or if the demand of the production changes. So the problem is not to just "find the minimum" but "assuming the minimum is obtained, what consequences can be deduced". In the final section 7, we provide conclusion and recommendations.

2. THE MATHEMATICAL MODEL

We consider that, for the fixed price, a competitive production firm is under contract to produce and deliver quantity Q units of a commodity during a specified time, say for instance, in a year, with the use of K quantity of capital, L quantity of labour, and R quantity of other inputs into its production process. These other inputs (e.g. land and other raw materials) are combined to produce the production (Humphery 1997). If the firm seeks to maximize its profit while meeting the terms of the contract, its production policy can be characterized as a constrained cost minimization problem in which the firm chooses the least cost combination of three factors: $K, L,$ and R to produce quantity Q units of the product (Baxley and Moorhouse 1984). To achieve its objective – the maximization of profit – the firm minimizes the cost function:

$$C\left(K, L, R\right) = rK + wL + \rho R, \tag{1}$$

subject to the constraint of production function:

$$Q = f\left(K, L, R\right), \tag{2}$$

where r is the rate of interest or services per unit of capital K, w is the wage rate per unit of labour L, and ρ is the cost per unit of other inputs R, while f is a suitable production function. A competitive production firm takes these and all factor prices as given. We assume that second order partial derivatives of the function f with respect to the independent variables (factors) K, L, and R exists.

Ignoring the actual form of the function C, we now formulate the minimization problem for the cost function given by (1) in terms of single Lagrange multiplier λ, by defining the Lagrangian function Z as follows:

$$Z\left(K, L, R, \lambda\right) = C\left(K, L, R\right) + \lambda\left(Q - f\left(K, L, R\right)\right). \tag{3}$$

This is a four dimensional unconstrained problem obtained from (1) and (2) by the use of Lagrange multiplier λ, as a device. Assuming that the competitive firm minimizes its cost, the optimal quantities $K^*, L^*, R^*, \text{and } \lambda^*$ of $K, L, R, \text{ and } \lambda$ that necessarily satisfy the first order conditions; which can be obtained by partially differentiating the Lagrangian function (3) with respect to four variables $\lambda,\ K,\ L,\ \text{and}\ R$ and setting them equal to zero:

$$Z_\lambda = Q - f\left(K, L, R\right) = 0, \tag{4a}$$

$$Z_K = C_K - \lambda f_K = 0, \tag{4b}$$

$$Z_L = C_L - \lambda f_L = 0, \tag{4c}$$

$$Z_R = C_R - \lambda f_R = 0, \tag{4d}$$

where,

$$Z_K = \frac{\partial Z}{\partial K},\ Z_L = \frac{\partial Z}{\partial L},\ Z_R = \frac{\partial Z}{\partial R},\ Z_\lambda = \frac{\partial Z}{\partial \lambda},$$

and $C_K = \dfrac{\partial C}{\partial K},\ C_L = \dfrac{\partial C}{\partial L},\ C_R = \dfrac{\partial C}{\partial R}$.

It may be noted that the partial derivative with respect to λ is just the same as the constraint - this is always the case, so we get again $Q = f\left(K, L, R\right)$, while from (4b-d), the Lagrange multiplier is obtained as follows:

$$\lambda = \frac{C_K}{f_K} = \frac{C_L}{f_L} = \frac{C_R}{f_R}. \tag{5}$$

Considering the infinitesimal changes $dK,\ dL,\ dR$ in $K,\ L,\ R$ respectively, and the corresponding changes dQ and dC, we get:

$$dC = C_K dK + C_L dL + C_R dR, \tag{6}$$

$$dQ = f_K dK + f_L dL + f_R dR. \tag{7}$$

With the use of (4b-d), or (5), we obtain the following equation:

$$\frac{dC}{dQ}=\frac{C_K dK + C_L dL + C_R dR}{f_K dK + f_L dL + f_R dR}=\lambda\,. \tag{8}$$

Thus, the Lagrange multiplier may be interpreted as the marginal cost of production; that is, it represents the increase in total costs incurred from the production of an additional unit Q. If, for example, one of the inputs, say K, is held constant, then (8) represents the partial derivative: $\left(\partial C/\partial Q\right)_K$, (with $dK=0$), and so.

3. AN EXPLICIT EXAMPLE

We now consider an explicit form of the production function f in (2), and provide a detailed discussion and intrinsic understanding of the problem at hand.
Let the function f is given by

$$Q=f\left(K,L,R\right)=AK^{\alpha}L^{\beta}R^{\gamma}, \tag{9}$$

where A is assumed to be unchanged technology; and the exponents $\alpha,\ \beta,$ and γ are the constants that constitute the output elasticities with respect to capital, labour, and other inputs (Humphery 1997) respectively. Using (1) and (9), (3) takes the following form:

$$Z\left(X,L,R,\lambda\right)=rK+wL+\rho R+\lambda\left(Q-AK^{\alpha}L^{\beta}R^{\gamma}\right). \tag{3a}$$

Therefore, (4a-d) become:

$$Z_{\lambda}=Q-AK^{\alpha}L^{\beta}R^{\gamma}=0\,, \tag{10a}$$

$$Z_K=r-\alpha\lambda AK^{\alpha-1}L^{\beta}R^{\gamma}=0\,, \tag{10b}$$

$$Z_L=w-\beta\lambda AK^{\alpha}L^{\beta-1}R^{\gamma}=0\,, \tag{10c}$$

$$Z_R=\rho-\gamma\lambda AK^{\alpha}L^{\beta}R^{\gamma-1}=0\,. \tag{10d}$$

Using the method of successful elimination and substitution, we solve above set of equations and obtain the optimum values of $K,\ L,\ R,$ and λ:

$$K=K^{*}=\frac{\alpha^{\left(\frac{\beta+\gamma}{\alpha+\beta+\gamma}\right)}w^{\left(\frac{\beta}{\alpha+\beta+\gamma}\right)}\rho^{\left(\frac{\gamma}{\alpha+\beta+\gamma}\right)}Q^{\left(\frac{1}{\alpha+\beta+\gamma}\right)}}{\beta^{\left(\frac{\beta}{\alpha+\beta+\gamma}\right)}\gamma^{\left(\frac{\gamma}{\alpha+\beta+\gamma}\right)}r^{\left(\frac{\beta+\gamma}{\alpha+\beta+\gamma}\right)}A^{\left(\frac{1}{\alpha+\beta+\gamma}\right)}}, \tag{11a}$$

$$L = L^{*} = \frac{\beta^{\left(\frac{\alpha+\gamma}{\alpha+\beta+\gamma}\right)} r^{\left(\frac{\alpha}{\alpha+\beta+\gamma}\right)} \rho^{\left(\frac{\gamma}{\alpha+\beta+\gamma}\right)} Q^{\left(\frac{1}{\alpha+\beta+\gamma}\right)}}{\alpha^{\left(\frac{\alpha}{\alpha+\beta+\gamma}\right)} \gamma^{\left(\frac{\gamma}{\alpha+\beta+\gamma}\right)} w^{\left(\frac{\alpha+\gamma}{\alpha+\beta+\gamma}\right)} A^{\left(\frac{1}{\alpha+\beta+\gamma}\right)}}, \tag{11b}$$

$$R = R^{*} = \frac{\gamma^{\left(\frac{\alpha+\beta}{\alpha+\beta+\gamma}\right)} r^{\left(\frac{\alpha}{\alpha+\beta+\gamma}\right)} w^{\left(\frac{\beta}{\alpha+\beta+\gamma}\right)} Q^{\left(\frac{1}{\alpha+\beta+\gamma}\right)}}{\alpha^{\left(\frac{\alpha}{\alpha+\beta+\gamma}\right)} \beta^{\left(\frac{\beta}{\alpha+\beta+\gamma}\right)} \rho^{\left(\frac{\alpha+\beta}{\alpha+\beta+\gamma}\right)} A^{\left(\frac{1}{\alpha+\beta+\gamma}\right)}}, \tag{11c}$$

$$\lambda = \lambda^{*} = \frac{r^{\left(\frac{\alpha}{\alpha+\beta+\gamma}\right)} w^{\left(\frac{\beta}{\alpha+\beta+\gamma}\right)} \rho^{\left(\frac{\gamma}{\alpha+\beta+\gamma}\right)} Q^{\left(\frac{1-\alpha-\beta-\gamma}{\alpha+\beta+\gamma}\right)}}{\alpha^{\left(\frac{\alpha}{\alpha+\beta+\gamma}\right)} \beta^{\left(\frac{\beta}{\alpha+\beta+\gamma}\right)} \gamma^{\left(\frac{\gamma}{\alpha+\beta+\gamma}\right)} A^{\left(\frac{1}{\alpha+\beta+\gamma}\right)}}. \tag{11d}$$

Thus, the stationary point is as below:

$$\left(K^{*}, L^{*}, R^{*}\right) = \left(\begin{array}{l} \dfrac{\alpha^{\left(\frac{\beta+\gamma}{\psi}\right)} w^{\left(\frac{\beta}{\psi}\right)} \rho^{\left(\frac{\gamma}{\psi}\right)} Q^{\left(\frac{1}{\psi}\right)}}{\beta^{\left(\frac{\beta}{\psi}\right)} \gamma^{\left(\frac{\gamma}{\psi}\right)} r^{\left(\frac{\beta+\gamma}{\psi}\right)} A^{\left(\frac{1}{\psi}\right)}}, \dfrac{\beta^{\left(\frac{\alpha+\gamma}{\psi}\right)} r^{\left(\frac{\alpha}{\psi}\right)} \rho^{\left(\frac{\gamma}{\psi}\right)} Q^{\left(\frac{1}{\psi}\right)}}{\alpha^{\left(\frac{\alpha}{\psi}\right)} \gamma^{\left(\frac{\gamma}{\psi}\right)} w^{\left(\frac{\alpha+\gamma}{\psi}\right)} A^{\left(\frac{1}{\psi}\right)}}, \\ \dfrac{\gamma^{\left(\frac{\alpha+\beta}{\psi}\right)} r^{\left(\frac{\alpha}{\psi}\right)} w^{\left(\frac{\beta}{\psi}\right)} Q^{\left(\frac{1}{\psi}\right)}}{\alpha^{\left(\frac{\alpha}{\psi}\right)} \beta^{\left(\frac{\beta}{\psi}\right)} \rho^{\left(\frac{\alpha+\beta}{\psi}\right)} A^{\left(\frac{1}{\psi}\right)}} \end{array} \right),$$

where $\psi = \alpha + \beta + \gamma.$ (12)

Moreover, substituting the values of K^{*}, L^{*}, R^{*} from (11a-c) into (1), we get the optimal value of the cost function in term of r, w, ρ, A, Q, α, β, and γ as follows:

$$C^{*} = (\alpha + \beta + \gamma) \left(\frac{r^{\left(\frac{\alpha}{\alpha+\beta+\gamma}\right)} w^{\left(\frac{\beta}{\alpha+\beta+\gamma}\right)} \rho^{\left(\frac{\gamma}{\alpha+\beta+\gamma}\right)} Q^{\left(\frac{1}{\alpha+\beta+\gamma}\right)}}{\alpha^{\left(\frac{\alpha}{\alpha+\beta+\gamma}\right)} \beta^{\left(\frac{\beta}{\alpha+\beta+\gamma}\right)} \gamma^{\left(\frac{\gamma}{\alpha+\beta+\gamma}\right)} A^{\left(\frac{1}{\alpha+\beta+\gamma}\right)}} \right). \tag{13}$$

4. INTERPRETATION OF LAGRANGE MULTIPLIER

Before we discuss sufficient conditions, we provide an interpretation of Lagrange multiplier, with the aid of chain rule, from (13) we get:

$$\frac{\partial C^*}{\partial Q} = C_K \frac{\partial K}{\partial Q} + C_L \frac{\partial L}{\partial Q} + C_R \frac{\partial R}{\partial Q}. \tag{14}$$

From (1), we get: $C_K = r$, $C_L = w$, $C_R = \rho$, and from (10b-d), we get:
$r = \alpha\lambda AK^{\alpha-1}L^{\beta}R^{\gamma}$, $w = \beta\lambda AK^{\alpha}L^{\beta-1}R^{\gamma}$, $\rho = \gamma\lambda AK^{\alpha}L^{\beta}R^{\gamma-1}$.

Therefore, we write (14) as follows:

$$\frac{\partial C^*}{\partial Q} = \lambda^*\left[\alpha AK^{\alpha-1}L^{\beta}R^{\gamma}\frac{\partial K}{\partial Q} + \beta AK^{\alpha}L^{\beta-1}R^{\gamma}\frac{\partial L}{\partial Q} + \lambda AK^{\alpha}L^{\beta}R^{\gamma-1}\frac{\partial R}{\partial Q}\right]. \tag{15}$$

From (10a), we have: $Q = AK^{\alpha}L^{\beta}R^{\gamma}$.
Differentiating above equation, keeping K, L, and R constants, we get:

$$1 = \alpha AK^{\alpha-1}L^{\beta}R^{\gamma}\frac{\partial K}{\partial Q} + \beta AK^{\alpha}L^{\beta-1}R^{\gamma-1}\frac{\partial L}{\partial Q} + \gamma AK^{\alpha}L^{\beta}R^{\gamma-1}\frac{\partial R}{\partial Q},$$

which allows us to rewrite (15) as:

$$\frac{\partial C^*}{\partial Q} = \lambda^*. \tag{16}$$

Therefore, (16) verifies (8). Thus, in this particular illustration, if the firm wants to increase (decrease) 1 unit of its production, it would cause total cost to increase (decrease) by approximately λ^* units, Lagrange multiplier is a shadow price. It is interesting to note that, unlike most of the cases where it is used, in this case the Lagrange multiplier has some sort of reasonable interpretation.

5. SUFFICIENT CONDITIONS

Now, in order to be sure that the optimal solution obtained in (12) is minimum; we check it against the sufficient conditions, which imply that for a solution K^*, L^*, R^*, and λ^* of (10a-d) to be a relative minimum, all the bordered principal minors of the following bordered Hessian,

$$|\overline{H}| = \begin{vmatrix} 0 & -Q_K & -Q_L & -Q_R \\ -Q_K & Z_{KK} & Z_{KL} & Z_{KR} \\ -Q_L & Z_{LK} & Z_{LL} & Z_{LR} \\ -Q_R & Z_{RK} & Z_{RL} & Z_{RR} \end{vmatrix},$$

should take the same sign, namely, the sign of $|\overline{H}_{m+1}|$ being that of $(-1)^m$, where m is number of constraints. It is important to note whether one has an odd or even number of constraints, for (-1) raised to an odd power will yield the opposite sign to the case of an even power. In our case $m=1$, therefore, all bordered principal minors should be negative. In other words, if

$$\left|\overline{H}_2\right| = \begin{vmatrix} 0 & -Q_K & -Q_L \\ -Q_K & Z_{KK} & Z_{KL} \\ -Q_L & Z_{LK} & Z_{LL} \end{vmatrix} < 0, \tag{17a}$$

and $$\left|\overline{H}_3\right| = \left|\overline{H}\right| = \begin{vmatrix} 0 & -Q_K & -Q_L & -Q_R \\ -Q_K & Z_{KK} & Z_{KL} & Z_{KR} \\ -Q_L & Z_{LK} & Z_{LL} & Z_{LR} \\ -Q_R & Z_{RK} & Z_{RL} & Z_{RR} \end{vmatrix} < 0, \tag{17b}$$

with all the derivatives evaluated at the critical values $K^*,\ L^*,\ R^*,$ and λ^*, then the stationary value of C obtained in (13) will assuredly be the minimum. We check this condition, through expanding first (17a):

$$\left|\overline{H}_2\right| = -Q_K Q_K Z_{LL} + 2Q_K Q_L Z_{KL} - Q_L Q_L Z_{KK}. \tag{18}$$

From (9) and (10b-d), we get:

$$Q_K = \alpha A K^{\alpha-1} L^{\beta} R^{\gamma};\ Q_L = \beta A K^{\alpha} L^{\beta-1} R^{\gamma};\ Q_R = \gamma A K^{\alpha} L^{\beta} R^{\gamma-1}. \tag{19a}$$

$$Z_{KK} = -\alpha(\alpha-1)\lambda A K^{\alpha-2} L^{\beta} R^{\gamma};\ Z_{LL} = -\beta(\beta-1)\lambda A K^{\alpha} L^{\beta-2} R^{\gamma};$$
$$Z_{RR} = -\gamma(\gamma-1)\lambda A K^{\alpha} L^{\beta} R^{\gamma-2}.$$

(19b)

$$Z_{KL} = Z_{LK} = -\alpha\beta\lambda A K^{\alpha-1} L^{\beta-1} R^{\gamma};\ Z_{KR} = Z_{RK} = -\alpha\gamma\lambda A K^{\alpha-1} L^{\beta} R^{\gamma-1};$$
$$Z_{LR} = Z_{RL} = -\beta\gamma\lambda A K^{\alpha} L^{\beta-1} R^{\gamma-1}. \tag{19c}$$

Substitution of the values of Q_K, Q_L, Z_{KK}, Z_{LL}, Z_{KL} from (19a-c) into (18) yields:

$$\left|\overline{H}_2\right| = -(\alpha+\beta)\alpha\beta\lambda A^3 K^{3\alpha-2} L^{3\beta-2} R^{3\gamma}.$$

Substitution of the critical values K^*, L^*, R^*, λ^* from (11a-d) into above equation, and after straightforward but tedious calculation yields:

$$\left|\overline{H}_2\right| = -(\alpha+\beta)\left(\frac{ r^{2\left(\frac{\beta+\gamma}{\psi}\right)} w^{2\left(\frac{\alpha+\gamma}{\psi}\right)} \gamma^{\left(\frac{3\gamma}{\psi}\right)} \alpha^{\left(\frac{2\alpha}{\psi}\right)} \beta^{\left(\frac{2\beta}{\psi}\right)} A^{\left(\frac{3}{\psi}\right)} }{ r^{\left(\frac{\alpha}{\psi}\right)} w^{\left(\frac{\beta}{\psi}\right)} \rho^{\left(\frac{3\gamma}{\psi}\right)} \alpha^{\left(\frac{\beta+\gamma}{\psi}\right)} \beta^{\left(\frac{\alpha+\gamma}{\psi}\right)} Q^{\left(\frac{3-2\psi}{\psi}\right)} } \right), \tag{20a}$$

where $\psi = \alpha + \beta + \gamma$.

Similarly, from (17b), we expand the determinant, noticing that the second partial derivative of $Z_{KL} = Z_{LK}$, $Z_{KR} = Z_{RK}$, and $Z_{LR} = Z_{RL}$, we get:

$$\begin{aligned}\left|\overline{H}\right| = & -Q_K Q_K Z_{LL} Z_{RR} + Q_K Q_K Z_{LR} Z_{LR} + 2\,Q_K Q_L Z_{KL} Z_{RR} - 2\,Q_K Q_R Z_{KL} Z_{LR} \\ & - 2\,Q_K Q_L Z_{KR} Z_{LR} + 2\,Q_K Q_R Z_{KR} Z_{LL} - Q_L Q_L Z_{KK} Z_{RR} + 2\,Q_L Q_R Z_{KK} Z_{LR} \\ & + Q_L Q_L Z_{KR} Z_{KR} - 2\,Q_L Q_R Z_{KL} Z_{KR} - Q_R Q_R Z_{KK} Z_{LL} + Q_R Q_R Z_{KL} Z_{KL}.\end{aligned}$$

Substituting the values of Q_K, Q_L, Q_R, Z_{KK}, Z_{LL}, Z_{RR}, Z_{KL}, Z_{KR}, Z_{LR} from (19a-c) into above equation, and after straightforward but tedious calculation, we get:

$$\left|\overline{H}\right| = -(\alpha+\beta+\gamma)\alpha\beta\gamma\lambda^2 A^4 K^{4\alpha-2} L^{4\beta-2} R^{4\gamma-2}.$$

Similarly, by substituting the critical values K^*, L^*, R^*, λ^* from (11a-d) into above equation, and after straightforward but tedious calculation, we get:

$$\left|\overline{H}\right| = -\psi\left(\frac{r^{2\left(\frac{\beta+\gamma}{\psi}\right)} w^{2\left(\frac{\alpha+\gamma}{\psi}\right)} \rho^{2\left(\frac{\alpha+\beta}{\psi}\right)} \alpha^{\left(\frac{3\alpha}{\psi}\right)} \beta^{\left(\frac{3\beta}{\psi}\right)} \gamma^{\left(\frac{3\gamma}{\psi}\right)} A^{\left(\frac{4}{\psi}\right)}}{r^{\left(\frac{2\alpha}{\psi}\right)} w^{\left(\frac{2\beta}{\psi}\right)} \rho^{\left(\frac{2\gamma}{\psi}\right)} \alpha^{\left(\frac{\beta+\gamma}{\psi}\right)} \beta^{\left(\frac{\alpha+\gamma}{\psi}\right)} \gamma^{\left(\frac{\alpha+\beta}{\psi}\right)} Q^2}\right),$$

where $\psi = \alpha + \beta + \gamma.$ (20b)

Since $A > 0,\ \alpha > 0,\ \beta > 0,\ \gamma > 0,$ and $r,\ w,\ \rho$ are the costs of inputs and hence are positive, while Q is production that will never be negative, therefore, from (20a) $\left|\overline{H}_2\right| < 0$ and from (20b) $\left|\overline{H}\right| < 0$, as required by (17a) and (17b), respectively. Equations (20a) and (20b) are sufficient conditions satisfied to state that the stationary point obtained in (12) is a relative minimum point. Thus, the value of the cost function obtained in (13) is indeed a relative minimum value.

6. COMPARATIVE STATIC ANALYSIS

Now, since sufficient conditions are satisfied, we drive further results of economic interest. Mathematically, we solve the four equations in (10a-d) for $K,\ L,\ R,$ and λ in terms of $r,\ w,\ \rho,$ and Q, and compute sixteen partial derivatives: $\frac{\partial \lambda}{\partial r}, \ldots,$ $\frac{\partial K}{\partial r}, \ldots,$ $\frac{\partial L}{\partial r}, \ldots,$ $\frac{\partial R}{\partial r}, \ldots,$ etc. These partial derivatives are referred to as the comparative static of the model. The model's usefulness is to determine how accurately it predicts the adjustments in the firm's input behaviour, that is, how the firm will react to the changes in the costs of capital, labour and other inputs. Since we have assumed that the left side of each equation in (10) is continuously differentiable and that the solution exists, then by the Implicit Function Theorem $K, L, R,$ and λ will each be continuously differentiable function of $r, w, \rho,$ and Q, if the following Jacobian matrix

$$J = \begin{bmatrix} 0 & -Q_K & -Q_L & -Q_R \\ -Q_K & Z_{KK} & Z_{KL} & Z_{KR} \\ -Q_L & Z_{LK} & Z_{LL} & Z_{LR} \\ -Q_R & Z_{RK} & Z_{RL} & Z_{RR} \end{bmatrix}, \tag{21}$$

is non-singular at the optimum point $(K^*, L^*, R^*, \lambda^*)$. As the sufficient conditions are met, so the determinant of (21) does not vanish at the optimum, that is, $|J| = |\overline{H}|$; consequently we apply the Implicit Function Theorem. Let F be the vector-valued function defined for the point $(\lambda^*, K^*, L^*, R^*, r, w, \rho, Q) \in R^8$, and taking the values in R^4, whose components are given by the left side of the equations in (10a-d). By the Implicit Function Theorem, the equation

$$F(\lambda^*, K^*, L^*, R^*, r, w, \rho, Q) = 0, \tag{22}$$

may be solved in the form of

$$\begin{bmatrix} \lambda^* \\ K^* \\ L^* \\ R^* \end{bmatrix} = G(r, w, \rho, Q). \tag{23}$$

Moreover, the Jacobian matrix for G is given by

$$\begin{bmatrix} \frac{\partial \lambda^*}{\partial r} & \frac{\partial \lambda^*}{\partial w} & \frac{\partial \lambda^*}{\partial \rho} & \frac{\partial \lambda^*}{\partial Q} \\ \frac{\partial K^*}{\partial r} & \frac{\partial K^*}{\partial w} & \frac{\partial K^*}{\partial \rho} & \frac{\partial K^*}{\partial Q} \\ \frac{\partial L^*}{\partial r} & \frac{\partial L^*}{\partial w} & \frac{\partial L^*}{\partial \rho} & \frac{\partial L^*}{\partial Q} \\ \frac{\partial R^*}{\partial r} & \frac{\partial R^*}{\partial w} & \frac{\partial R^*}{\partial \rho} & \frac{\partial R^*}{\partial Q} \end{bmatrix} = -J^{-1} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{24}$$

where the ith row in the last matrix on the right is obtained by differentiating the ith left side in (10) with respect to r, then w, then ρ, and then Q. Let C_{ij} be the cofactor of the element in the ith row and jth column of J, and then inverting J using the method of cofactor gives: $J^{-1} = \frac{1}{|J|}C^T$, where $C = (C_{ij})$. Thus, following matrix multiplication rule, (24) can further be expressed in the following form:

$$\begin{bmatrix} \frac{\partial \lambda^*}{\partial r} & \frac{\partial \lambda^*}{\partial w} & \frac{\partial \lambda^*}{\partial \rho} & \frac{\partial \lambda^*}{\partial Q} \\ \frac{\partial K^*}{\partial r} & \frac{\partial K^*}{\partial w} & \frac{\partial K^*}{\partial \rho} & \frac{\partial K^*}{\partial Q} \\ \frac{\partial L^*}{\partial r} & \frac{\partial L^*}{\partial w} & \frac{\partial L^*}{\partial \rho} & \frac{\partial L^*}{\partial Q} \\ \frac{\partial R^*}{\partial r} & \frac{\partial R^*}{\partial w} & \frac{\partial R^*}{\partial \rho} & \frac{\partial R^*}{\partial Q} \end{bmatrix} = -\frac{1}{|J|}\begin{bmatrix} C_{21} & C_{31} & C_{41} & C_{11} \\ C_{22} & C_{32} & C_{42} & C_{12} \\ C_{23} & C_{33} & C_{43} & C_{13} \\ C_{24} & C_{34} & C_{44} & C_{14} \end{bmatrix}. \qquad (25)$$

Now, we study the effects of changes in r, w, ρ, and Q on K, L, and R. Firstly, we find out the effect on capital K when it's interest rate r increases. From (25), we get:

$$\frac{\partial K^*}{\partial r} = -\frac{1}{|J|}[C_{22}] = -\frac{1}{|J|}\begin{vmatrix} 0 & -Q_L & -Q_R \\ -Q_L & Z_{LL} & Z_{LR} \\ -Q_R & Z_{RL} & Z_{RR} \end{vmatrix}.$$

Expansion of above determinant yields:

$$\frac{\partial K^*}{\partial r} = -\frac{1}{|J|}\left\{-Q_L Q_L Z_{RR} + 2\,Q_L Q_R Z_{LR} - Q_R Q_R Z_{LL}\right\}.$$

Substituting the values of Q_L, Q_R, Z_{LL}, Z_{RR}, and Z_{LR} from (19a-c) into the above equation, and after straightforward calculation, we get:

$$\frac{\partial K^*}{\partial r} = \frac{1}{|J|}(\beta + \gamma)\beta\gamma\lambda A^3 K^{3\alpha} L^{3\beta-2} R^{3\gamma-2}.$$

Since $|J| = |\overline{H}|$, therefore, by substituting the value of $|\overline{H}|$ from (20b), as well as the optimal values of K^*, L^*, R^*, λ^* from (11a-d) into the above equation, and after straightforward but tedious calculation, we get:

$$\frac{\partial K^*}{\partial r} = -\frac{(\beta+\gamma)}{(\alpha+\beta+\gamma)}\left[\frac{\alpha^{\left(\frac{\beta+\gamma}{\alpha+\beta+\gamma}\right)} w^{\left(\frac{\beta}{\alpha+\beta+\gamma}\right)} \rho^{\left(\frac{\gamma}{\alpha+\beta+\gamma}\right)} Q^{\left(\frac{1}{\alpha+\beta+\gamma}\right)}}{r^{\left(\frac{\alpha+2\beta+2\gamma}{\alpha+\beta+\gamma}\right)} \beta^{\left(\frac{\beta}{\alpha+\beta+\gamma}\right)} \gamma^{\left(\frac{\gamma}{\alpha+\beta+\gamma}\right)} A^{\left(\frac{1}{\alpha+\beta+\gamma}\right)}}\right].$$

Since $\alpha > 0,\ \beta > 0,\ \gamma > 0,\ A > 0,$ and $r > 0,\ w > 0,\ \rho > 0,$ and Q is the output of the firm that can never be negative, therefore,

$$\frac{\partial K^*}{\partial r} < 0, \tag{26}$$

which indicates that if the interest rate or services of the capital K increases, the firm may consider decreasing the level of input K.

Secondly, we examine the effects on labour L when the interest rate of capital K increases. Again from (25), we get:

$$\frac{\partial L^*}{\partial r} = -\frac{1}{|J|}[C_{23}] = \frac{1}{|J|}\begin{vmatrix} 0 & -Q_K & -Q_R \\ -Q_L & Z_{LK} & Z_{LR} \\ -Q_R & Z_{RK} & Z_{RR} \end{vmatrix}.$$

$$\frac{\partial L^*}{\partial r} = \frac{1}{|J|}\{-Q_K Q_L Z_{RR} + Q_K Q_R Z_{LR} + Q_L Q_R Z_{RK} - Q_R Q_R Z_{LK}\}.$$

By substituting the values of Q_K, $Q_{L,}$ Q_R, Z_{RR}, Z_{LR}, $Z_{RK,}$ and Z_{LK} from (19a-c) into the above equation, and after simplification, we get:

$$\frac{\partial L^*}{\partial r} = -\frac{1}{|J|}\alpha\beta\gamma\lambda A^3 K^{3\alpha-1} L^{3\beta-1} R^{3\gamma-2}.$$

Since $|J| = |\overline{H}|$, therefore, by putting the value of $|\overline{H}|$ from equation (20b), as well as the optimal values of K^*, L^*, R^*, λ^* from (11a-d), and after straightforward but tedious calculation, we get:

$$\frac{\partial L^*}{\partial r} = \frac{1}{(\alpha+\beta+\gamma)}\left(\frac{\alpha^{\left(\frac{\beta+\gamma}{\alpha+\beta+\gamma}\right)} \beta^{\left(\frac{\alpha+\gamma}{\alpha+\beta+\gamma}\right)} \rho^{\left(\frac{\gamma}{\alpha+\beta+\gamma}\right)} Q^{\left(\frac{1}{\alpha+\beta+\gamma}\right)}}{r^{\left(\frac{\beta+\gamma}{\alpha+\beta+\gamma}\right)} w^{\left(\frac{\alpha+\gamma}{\alpha+\beta+\gamma}\right)} \gamma^{\left(\frac{\gamma}{\alpha+\beta+\gamma}\right)} A^{\left(\frac{1}{\alpha+\beta+\gamma}\right)}}\right).$$

Again, since $\alpha > 0,\ \beta > 0,\ \gamma > 0, A > 0,$ and $r > 0,\ w > 0,\ \rho > 0,$ and Q is the output of the firm that can never be negative, therefore,

$$\frac{\partial L^*}{\partial r} > 0, \tag{27}$$

which indicates that when the interest rate or services of capital K increases the firm can increase the level of labour L, because both inputs are unrelated to each other, as $C_{KL} = 0$. In other words, in the context of the present problem of the firm's cost minimization, where the firm produces output Q from the inputs of capital K, labour L and other inputs R, so (27) supports the common sense that both inputs neither complement nor supplement, but they are unrelated.

The above analysis relates to the effects of a change in interest rate of capital K; our results are readily adaptable to the case of a change in wage rate of labour L, as well as to a change in cost of other inputs R.

Next, we analyze the effect of a change in output Q. Suppose the firm gets an additional order of its product to produce and supply, so it wants to increase it's output Q, then naturally, we can expect that there will an increase in its inputs $K,\ L,$ and R. We examine and verify this mathematically as follows. From (25), we get:

$$\frac{\partial K^*}{\partial Q} = -\frac{1}{|J|}[C_{12}] = \frac{1}{|J|}\begin{vmatrix} -Q_K & Z_{KL} & Z_{KR} \\ -Q_L & Z_{LL} & Z_{LR} \\ -Q_R & Z_{RL} & Z_{RR} \end{vmatrix}.$$

$$\frac{\partial K^*}{\partial Q} = \frac{1}{|J|}\begin{Bmatrix} -Q_K Z_{LL} Z_{RR} + Q_K Z_{LR} Z_{RL} + Q_L Z_{KL} Z_{RR} - Q_R Z_{KL} Z_{LR} \\ -Q_L Z_{KR} Z_{RL} + Q_R Z_{LL} Z_{KR} \end{Bmatrix}.$$

By substituting the values of $Q_K,\ Q_{L,}\ Q_R,\ Z_{LL,}\ Z_{RR},\ Z_{LR},\ Z_{RK,}$ and Z_{LK} from (19a-c) into the above equation, and after simplification, we get:

$$\frac{\partial K^*}{\partial Q} = -\frac{1}{|J|}\alpha\beta\gamma\lambda^2 A^3 K^{3\alpha-1} L^{3\beta-2} R^{3\gamma-2}.$$

Again, since $|J| = |\overline{H}|$, therefore, by putting the value of $|\overline{H}|$ from (20b), as well as the optimal values of K^*, L^*, R^*, λ^* from (11a-d), and after straightforward but tedious calculation, we get:

$$\frac{\partial K^*}{\partial Q} = \frac{1}{(\alpha+\beta+\gamma)}\left[\frac{\alpha^{\left(\frac{\beta+\gamma}{\alpha+\beta+\gamma}\right)} w^{\left(\frac{\beta}{\alpha+\beta+\gamma}\right)} \rho^{\left(\frac{\gamma}{\alpha+\beta+\gamma}\right)} Q^{\left(\frac{1-\alpha-\beta-\gamma}{\alpha+\beta+\gamma}\right)}}{r^{\left(\frac{\beta+\gamma}{\alpha+\beta+\gamma}\right)} \beta^{\left(\frac{\beta}{\alpha+\beta+\gamma}\right)} \gamma^{\left(\frac{\gamma}{\alpha+\beta+\gamma}\right)} A^{\left(\frac{1}{\alpha+\beta+\gamma}\right)}}\right].$$

Again, since $\alpha > 0,\ \beta > 0,\ \gamma > 0, A > 0,$ and $r > 0,\ w > 0,\ \rho > 0,$ and Q is output of the firm that can never be negative, therefore,

$$\frac{\partial K^*}{\partial Q} > 0, \tag{28}$$

which verifies our assumption and common sense that when the demand of the product increases the firm may consider increasing it's level of inputs: capital, labour, and other inputs.

7. CONCLUSION AND RECOMMENDATIONS

In this paper, Lagrange multiplier method is applied to a firm's cost minimization problem subject to Cobb-Douglas production function as an output constraint. An attempt is made to apply necessary and sufficient conditions for optimal values – in this particular case, minimization of the cost of a firm. It is demonstrated that the value of the Lagrange multiplier is positive. Unlike most of the cases where it is used, in this particular illustration, a reasonable interpretation of the Lagrange multiplier is presented, that is, if the firm wants to increase (decrease) 1 unit of its production, it would cause total cost to increase (decrease) by approximately λ^* units, Lagrange multiplier is a shadow price. With the help of comparative static results and the application of Implicit Function Theorem, we mathematically showed the behavior of the firm, and recommend that if the cost of a particular input increases, the firm needs to consider decreasing the level of that particular input; at the same time, and there is no effect on the level of other inputs. As well as, we demonstrated mathematically that when the demand of the product increases the firm may consider increasing its level of inputs: capital, labour, and other inputs.

REFERENCES

Baxley, J.V and J. C. Moorhouse. 1984. Lagrange Multiplier Problems in Economics. *The American Mathematical Monthly*, 91(7): 404-412 (Aug. – Sep)

Chiang, A.C. 1984. *Fundamental Methods of Mathematical Economics*. 3rd ed. Singapore: McGraw-Hill

Humphery, T.M. 1997. Algebraic Production Functions and Their Uses before Cobb-Douglas. *Federal Reserve Bank of Richmond Economic Quarterly*, 83 (1) winter.

Islam, J.N. (Ed.).1997. *Aspects of Mathematical Economics and Social Choice Theory*. Proceedings of the Second Chittagong Conference on Mathematical Economics and its Relevance for Development. Chittagong: University of Chittagong, Bangladesh.

Pahlaj M. 2002. *Theory and Applications of Classical Optimization to Economic Problems.* M. Phil. Thesis. Chittagong: University of Chittagong, Bangladesh.

***